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# BUCKLING BEHAVIOUR OF IMPERFECT ELASTIC AND LINEARLY VISCOELASTIC STRUCTURES

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**Abstract**—In a previous paper, the author has shown that the behaviour of imperfect elastic structures subjected to buckling forces could be predicted on the basis of the eigenvalues and eigenmodes. After a brief recall of these properties, it is first shown—in Appendix—that they extend to cases of spatial buckling like the buckling of flexure and torsion. Then, it is shown that the correspondence principle valid in first-order behaviour of linearly viscoelastic structures can be extended in full generality—by the use of Laplace transformation—to buckling problems, whatever be the constitutive equations of the material.

Finally, several examples of eulerian buckling or flexural–torsional buckling of a bar and of buckling of a plate, are treated in detail.

## 1. INTRODUCTION

In 1940, the author has devoted a long paper [1] to the study of the stability of elastic structures and, in particular, to the behaviour of structures presenting slight imperfections with regard to the ideal shape for which they would buckle by bifurcation of the equilibrium. His main results have been reproduced in his book [2].

The aim of this paper is: first, to deduce the results obtained previously, starting directly from the general finite displacement theory, and to show that they are applicable to cases of spatial instability like flexural torsional buckling; secondly, to present an extension of these results to imperfect linearly viscoelastic structures, with some practical applications to the instability of bars subjected to eulerian buckling or to flexural–torsional buckling and of plates loaded in their plane.

## 2. EIGENVALUES, EIGENMODES AND THEIR PROPERTIES FOR ELASTIC STRUCTURES

We follow the analysis of Washizu [3] and use the finite displacement elasticity referred to the original axes (Lagrangian coordinates). The coordinates are represented by greek suffixes: $x^\alpha$, $x^\beta$, $x^\gamma$. In its primary stage, the structure is subjected to body forces $k\bar{P}^{(0)\lambda}$ in $V$ and surface tractions $k\bar{F}^{(0)\lambda}$ on $S_T$, and $u^\lambda = 0$ on $S_u \equiv S - S_T$, where $k$ is a monotonically increasing factor of proportionality, called the *multiplier*. We employ as criterion of instability the existence, for the value $k_{cr}$ of the multiplier, of an adjacent equilibrium configuration.

Calling $\sigma^{\lambda\mu}$ the additional stresses caused by the infinitely small virtual displacement field added to the field of the primary stresses $k^{(0)\lambda\mu}$, we obtain by the energy criterion of stability (deduced from the principle of virtual work) that the condition of indifferent equilibrium is

$$\delta\Pi = 0 \quad \text{and} \quad \delta^2\Pi = 0 = \text{minimum or } \delta(\delta^2\Pi) = 0 \tag{2.1}$$

where $\Pi$ is the potential energy of the structure.

With the notation

$$\partial\alpha^{u} \equiv \frac{\partial u}{\partial x^{\alpha}}, \tag{2.2}$$

the expression (2.1) may be written explicitely, [3]

$$\iiint_v (\sigma^{\lambda\mu}\delta\varepsilon_{\lambda\mu} + k\sigma^{(0)\lambda\mu}\,\partial_\lambda u^\kappa\,\partial_\mu \delta u^\kappa)\mathrm{d}V = 0 \tag{2.3}$$

with the boundary conditions

$$\sigma^{\lambda\kappa}n_\kappa + k\sigma^{(0)\kappa\mu}n_\kappa\,\partial_\mu u^\lambda = 0 \text{ on } S_T \tag{2.4}$$

$$u^\lambda = 0 \text{ on } S_u. \tag{2.5}$$

In (2.3), use has been made of the summation convention for the repeated subscripts $\lambda$, $\mu$.

The additional stresses and strains produced by the additional displacements conducing to the secondary shape are assumed to be connected by relations of the form

$$\sigma^{\lambda\mu} = a^{\lambda\mu\alpha\beta}\varepsilon_{\alpha\beta} \tag{2.6}$$

satisfying the symmetry relation $a^{\lambda\mu\alpha\beta} = a^{\alpha\beta\lambda\mu}$. There exists therefore a strain energy

$$A(\varepsilon_{\lambda\mu};\,\sigma^{(0)\lambda\mu}) = \tfrac{1}{2}a^{\lambda\mu\alpha\beta}\varepsilon_{\lambda\mu}\varepsilon_{\alpha\beta} \tag{2.7}$$

and the principle (2.3) may be written

$$\delta^2\Pi = \iiint_v \{A(u^\lambda;\,\sigma^{(0)\lambda\mu}) + \tfrac{1}{2}k\sigma^{(0)\lambda\mu}\,\partial_\lambda u^\kappa\,\partial_\mu u^\kappa\}\,\mathrm{d}V = 0 = \text{minimum}. \tag{2.8}$$

Putting

$$\iiint_v A\,\mathrm{d}V = \delta^2 U \tag{2.9}$$

and

$$-\frac{1}{2}\iiint_v \sigma^{(0)\lambda\mu}\,\partial_\lambda u^\kappa\,\partial_\mu u^\kappa\,\mathrm{d}V = \delta^2 T, \tag{2.10}$$

we may write (2.8) as follows:

$$\delta^2\Pi = \delta^2 U + \delta^2 V, \tag{2.11}$$

where

$$\delta^2 V = -k\delta^2 T \tag{2.12}$$

and $\delta^2 U$ are the variations, during buckling, of the potential energy of the external forces and of the strain energy, respectively.

$\delta^2 T$ is, like $\delta^2 U$, a quadratic and homogeneous functional of the additional displacements. Equation (2.1) may therefore be written

$$k_{cr} = -\frac{\iiint_v A(u^\lambda)\,\mathrm{d}V}{\frac{1}{2}\iiint \sigma^{(0)\lambda\mu}\,\partial_\lambda u^\kappa\,\partial_\mu u^\kappa\,\mathrm{d}V} = \frac{\delta^2 U(u^\lambda)}{\delta^2 T(u^\lambda)} = \text{minimum}, \tag{2.13}$$

which is the celebrated Rayleigh's principle adapted to buckling problems.

We shall admit that, for the structures we consider, the variational problem (2.13) possesses effectively a solution. Then, it can be shown [2] that the structure enters in indifferent equilibrium for a series of values of $k$, called *instability eigenvalues,*

$$k_1, k_2, \ldots, k_n \ldots.$$

and that it then takes configurations $u_1^\lambda(x^\alpha)$, $u_2^\lambda(x^\alpha)$, ... associated to these loads, that we shall call *instability eigenmodes.*

To give a concrete and familiar example of foregoing considerations, we consider a prismatic simply supported bar subjected to an axial thrust $P$. In that case, the Rayleigh quotient becomes (Fig. 1).

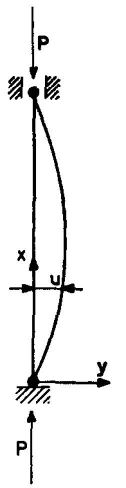


Fig. 1. Buckling of an ideally straight bar axially compressed.

$$P_{cr} = \frac{\delta^2 U(u)}{\delta^2 T(u)} = \frac{\dfrac{EI}{2}\displaystyle\int_0^l u''^2 \, dx}{\dfrac{1}{2}\displaystyle\int_0^l u'^2 \, dx} \tag{2.14}$$

the instability eigenvalues (critical loads) are

$$P_n = \frac{n^2 \pi^2 EI}{l^2} \tag{2.15}$$

and the eigenmodes (buckling shapes) are

$$u_n = A_n \sin \frac{n\pi x}{l} \tag{2.16}$$

To simplify the notations, $\delta^2 U$, $\delta^2 V$ and $\delta^2 T$ will, in what follows, be replaced by $U$, $V$ and $T$ respectively. It is convenient to normalize the eigenmodes, that means to choose them so as to have

$$T(u_n) = 1 \tag{2.17}$$

and therefore by (2.13)

$$U(u_n) = k_n. \tag{2.18}$$

In the particular case of the compressed bar, this is obtained by choosing

$$u_n = \frac{2\sqrt{l}}{n\pi} \sin \frac{n\pi x}{l}. \tag{2.19}$$

An important property of the eigenmodes is that *they are orthogonal with respect to the two functionals U and T*. That means that, for any two different modes $u_i$, $u_j$, we have

$$U_{11}(u_i, u_j) = 0 \quad \text{and} \quad T_{11}(u_i, u_j) = 0 \qquad (i \neq j) \tag{2.20}$$

where $U_{11}$ and $T_{11}$ are the two bilinear functionals associated to $U$ and $T$, respectively. For example, in the particular case of Fig. 1, we have

$$U_{11}(u_i, u_j) \equiv \frac{1}{2}\int_0^l EIu_i''u_j''\,dx = 0 \quad \text{and} \quad T_{11} \equiv \frac{1}{2}\int_0^l u_i'u_j'\,dx = 0. \tag{2.21}$$

As a result of this property, any displacement field of the structure satisfying the geometrical boundary conditions of the problem may be expanded in a series of normalized instability modes as follows

$$u^{(\lambda)} = \sum_{n=1}^{\infty} a_n u_n^{\lambda}. \tag{2.22}$$

It is easy to verify then that $U(u^\lambda)$ and $T(u^\lambda)$ take the form

$$U(u^\lambda) = \sum_{n=1}^{\infty} k_n a_n^2 \tag{2.23}$$

$$T(u^\lambda) = \sum_{n=1}^{\infty} a_n^2 \tag{2.24}$$

If we call *principal coordinates* of a displacement field the numbers $a_n$ defined by (2.22), we have the theorem:
*If the considered structure possesses an infinity of instability modes forming a complete system, then the variation, during buckling, of the strain energy, U, and of the energy of external forces, T, may be expressed as sums of the squares of the principal coordinates.*

## 3. BEHAVIOUR OF IMPERFECT ELASTIC STRUCTURES SUBJECTED TO INSTABILITY

Actual structures never possess the ideal shape that we have implicitely assumed in Section 2. For example, in the simple case of the axially compressed bar, the bar axis is never rigorously straight and the line of action of the thrust does never coincide rigorously with this axis.

In what follows, we call, in order to simplify, *structure* the actual imperfect structure and *model* the perfect reference structure. Our aim is to show that the behaviour of the structure may be deduced simply from the behaviour of the model.

In the developments which follow, we do not consider only these displacements, but also other quantities like the rotations of cross sections, the curvature of the axis, the bending moment in a certain section, etc. which vary in proportion to these displacements $u$. To simplify the language, we designate all these quantities by the general term "effects", and we represent them by the notation $F$.

The characteristic of the (imperfect) structure is to present, in its unloaded shape, small initial displacements $u_0$ with respect to the perfect model. These initial displacements may always be represented by an expansion in series of instability modes of the form

$$u_0 = \sum_{n=1}^{\infty} a_n^0 u_n . \tag{3.1}$$

Let us now apply to this structure a set of forces $kP_i$, capable to cause its instability. If the *additional* displacements produced by these forces are called $u$, they may equally well be represented by a similar series

$$u = \sum_{n=1}^{\infty} a_n u_n . \tag{3.2}$$

An effect $F_0$, whatsoever, which is produced in the structure, may be represented by a similar development

$$F_0 = \sum_{n=1}^{\infty} F_n u_n \tag{3.3}$$

where the $F_n$ are called *principal effects*.
According to (2.20) and (3.2), the variation of the strain energy of the structure when the buckling forces $kP_i$ are applied, is

$$U(u) = \sum_n k_n a_n^2 \tag{3.4}$$

because the structure has a so small imperfection $u_0$ that its strain energy, expressed in the *additional* displacements, is the same as that of the adjacent perfect model.

On the other hand, the variation of the potential energy of the structure when the external forces $kF_i$ are applied, may be evaluated by giving to this imperfect structure the displacements $(-u_0)$ in order to let it coincide with the perfect model, then by giving to this model the deformation $(u_0 + u)$. In view of the relation

$$V = -kT \tag{2.12}$$

and of

$$T(u) = \sum_{n=1}^{\infty} a_n^2, \tag{2.22}$$

we find

$$V(u) = -k\left[\sum_n (a_n^0 + a_n)^2 - \sum_n (a_n^0)^2\right] . \tag{3.5}$$

The structure being in equilibrium, we may apply the principle of virtual work, according to which

$$\delta(U + V) = \delta\left\{\sum k_n a_n^2 - k\left[\sum_n (a_n^0 + a_n)^2 - \sum_n (a_n^0)^2\right]\right\} = 0. \tag{3.6}$$

Adopting for virtual deformation the instability mode $u_n$ of index $n$, we have:

$$\delta(U+V) = \frac{\partial(U+V)}{\partial a_n} = 2(k_n a_n - k a_n + k a_n^0) = 0$$

which yields

$$a_n = \frac{k a_n^0}{k_n - k} = \frac{a_n^0}{k_n/k - 1} \quad (n = 1, 2, \ldots). \tag{3.7}$$

The additional deformation of the structure is by (3.2)

$$u = \sum_n \frac{1}{(k_n/k) - 1} a_n^0 u_n \tag{3.8}$$

and the total deformation of the structure measured from the perfect configuration of the model is therefore

$$u_t = u_0 + u = \sum_n \frac{1}{1 - k/k_n} a_n^0 u_n. \tag{3.9}$$

If, instead of the displacement "$u$" an "effect" $F$ whatsoever produced in the structure is considered, fully similar results

$$F = \sum_n \frac{1}{k_n/k - 1} F_n^0 u_n \tag{3.10}$$

and

$$F_t = F_0 + F = \sum_n \frac{1}{1 - k/k_n} F_n^0 u_n \tag{3.11}$$

are obtained.

All the results above may be cast into the following theorem:

*Theorem* 1. *The effect of buckling forces is to increase every principal coordinate $a_n^0$ describing the initial deformation of the imperfect structure in the ratio*

$$\frac{1}{k_n/k - 1},$$

*where $k$ represents the intensity of the buckling forces and $k_n$ their critical intensity producing the buckling of the model in its* $n$th *mode. All the "principal effects" $F_n^0$ produced in the structure are increased in the same proportion.*

Similar considerations [1, 2], may be developed for perfect structures which are loaded by "*ordinary*" forces, that means forces like $Q$ (Fig. 2) which cannot produce the instability of the structure, and the two following theorems may be established:

*Theorem* 2. *An elastic structure liable to buckling has the same behaviour whether its initial deformation before application of the buckling forces $kP_i$ be a natural deformation or a deformation produced by "ordinary" forces.*

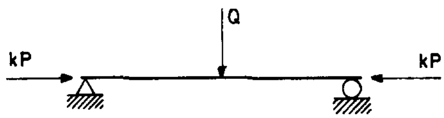


Fig. 2. Bar subjected to "ordinary" forces.

*Theorem* 3. (*Generalized principle of superposition*): *If, in an elastic structure subjected to external forces, the buckling forces* $kP_i$ *remain constant, the deformation resulting from various groups of ordinary forces may be obtained by superposing the deformations produced separately by each of these groups acting together with the buckling forces.*

*Important remark*

In the course of this paragraph, the displacement of a point of the structure has been treated as a scalar quantity, $u$, whereas it is clear from paragraph 2 that it must be a vectorial quantity $u^\lambda$ in the cases of spatial instabilities. We shall show in Appendix that the properties of the eigenmodes and eigenvalues recalled in Section 2 and those of the additional displacements $u$ of imperfect structures established in Section 3 are still valid in the case of flexural torsional buckling.

# II—LINEARLY VISCOELASTIC STRUCTURES

## 4. CONSTITUTIVE EQUATIONS AND CORRESPONDENCE PRINCIPLE

Clear accounts of the theory of linearly viscoelastic materials are available [4–9]. In the most general anisotropic case, the stresses and strains in these materials are related by means of the Boltzmann superposition integral

$$\sigma_{ij} = \int_0^t C_{ij}^{kl}(t-\tau)\frac{\partial \varepsilon_{kl}}{\partial \tau}\,\mathrm{d}\tau, \tag{4.1}$$

where the symmetry of the stress and strain tensors implies the relations

$$C_{ij}^{kl}(t) = C_{ji}^{kl}(t) = C_{ji}^{lk}(t). \tag{4.2}$$

In applications, it is often desirable to use the inverse of constitutive equation (4.1):

$$\varepsilon_{ij} = \int_0^t S_{ij}^{kl}(t-\tau)\frac{\partial \sigma_{kl}}{\partial \tau}\,\mathrm{d}\tau \tag{4.3}$$

$C_{ij}^{kl}(t)$ and $S_{ij}^{kl}(t)$ are components of fourth order tensors and are called *relaxation modulus* and *creep compliance*, respectively.

Introducing the Laplace transforms of above quantities

$$\bar{f} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = L|f(t)| \tag{4.4}$$

and transforming equations (4.2) and (4.3) by means of the rule for convolution integrals

$$\bar{f}(s)\bar{g}(s) = L\left\{\int_0^t f(t-\tau)g(\tau)\,d\tau\right\} \tag{4.5}$$

yields the algebraic relations

$$\bar{\sigma}_{ij} = \tilde{C}_{ij}^{kl}\bar{\varepsilon}_{kl} \tag{4.6}$$

$$\bar{\varepsilon}_{ij} = \tilde{S}_{ij}^{kl}\bar{\sigma}_{kl}, \tag{4.7}$$

where $\tilde{C}_{ij}^{kl}$ and $\tilde{S}_{ij}^{kl}$ are defined as $s$ – multiplied transforms (also called Carson transforms) of the relaxation moduli and creep compliances:

$$\tilde{C}_{ij}^{kl} \equiv s\bar{C}_{ij}^{kl}, \qquad \tilde{S}_{ij}^{kl} \equiv \bar{S}\,\|_{ij}^{kl}. \tag{4.8}$$

The classical equations of the theory of elasticity read:

$$\partial_j\sigma_{ij} + F_i = 0 \tag{4.9}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i) \tag{4.10}$$

with the boundary conditions

$$u_i = U_i \text{ on } S_u \tag{4.11}$$

and

$$\sigma_{ij}u_j = T_i \text{ on } S_T \tag{4.12}$$

whereas Hooke's law is, in the general anisotropic case,

$$\sigma_{ij} = C_{ij}^{kl}\varepsilon_{kl}. \tag{4.13}$$

Taking the Laplace transforms of equations (4.9–4.12), we obtain

$$\partial_j\bar{\sigma}_{ij} + \bar{F}_i = 0 \tag{4.14}$$

$$\bar{\varepsilon}_{ij} = \tfrac{1}{2}(\partial_i\bar{u}_j + \partial_j\bar{u}_i) \tag{4.15}$$

$$\bar{u}_i = \bar{U}_i \text{ on } S_u \tag{4.16}$$

and

$$\bar{\sigma}_{ij}n_j = \bar{T}_i \text{ on } S_T. \tag{4.17}$$

If we add to these equations formulae (4.6) and (4.7) which replace (4.13), we see that: *Correspondence principle. The basic equations of linearly viscoelastic materials are formally equivalent to those of an "associated" elastic problem for the same geometric body subjected to imposed displacement* $U_i = U_i(x_j, s)$, *surface tractions* $\bar{T}_i = T_i(x_j, s)$ *and body forces* $\bar{F}_i = \bar{F}_i(x_i, s)$.

The equations (4.10) and (4.15) are only valid for very small displacements, so that the validity of the correspondence principle is so far restricted to first order problems. It is the main aim of present paper to extend it for second-order theory problems such as buckling problems.

It is easily shown that the correspondence principles extends to all variational methods of elastic analysis. For our present purpose, it will suffice to show this for the principle of potential energy. In elastic analysis, this principle reads

$$\Pi \equiv \iiint_v U \,\mathrm{d}V - \iiint_v F_i u_i \,\mathrm{d}V - \iint_{S_T} T_i u_i \,\mathrm{d}S = 0 = \min \tag{4.18}$$

where

$$U = \tfrac{1}{2} C_{ij}^{kl} \varepsilon_{ij} \varepsilon_{kl} \tag{4.19}$$

is the strain energy density.

For the associated linearly viscoelastic body, the principle reads

$$\bar{\Pi} \equiv \iiint_v \bar{U} \,\mathrm{d}V - \iiint_v \bar{F}_i \bar{u}_i \,\mathrm{d}V - \iint_{S_T} \bar{T}_i \bar{u}_i \,\mathrm{d}S = 0 = \text{minimum} \tag{4.20}$$

where

$$\bar{U} = \tfrac{1}{2} \tilde{C}_{ij}^{kl} \bar{\varepsilon}_{ij} \bar{\varepsilon}_{kl} \tag{4.20 bis}$$

is the "associated strain energy density".

## 5. CONSTITUTIVE EQUATIONS FOR ISOTROPIC VISCOELASTIC MATERIALS

In the isotropic case, the general Hooke's law (4.13) reduces to

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{ii} + 2\mu \varepsilon_{ij} \tag{5.1}$$

where $\varepsilon_{ij}$ is the Kronecker delta.

The Lamé coefficients $\lambda$ and $\mu$ are related to the engineering constants $E$, $\nu$, by the familiar relations

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \mu = G = \frac{E}{2(1+\nu)}. \tag{5.2}$$

In view of the preceding developments, the law replacing (5.1) for an isotropic viscoelastic material reads

$$\bar{\sigma}_{ij} = \tilde{\lambda} \delta_{ij} \bar{\varepsilon}_{ii} + 2\tilde{\mu} \bar{\varepsilon}_{ij} \tag{5.3}$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ are the $s$-multiplied Laplace transforms (also called Carson transforms) of $\lambda$ and $\mu$.

The stress tensor may be decomposed in a spherical tensor $s \equiv (\sigma_{kk}/3)\delta_{ij}$ and a deviator $s_{ij}$ as follows

$$\sigma_{ij} = \frac{\sigma_{kk}}{3} \delta_{ij} + s_{ij}. \tag{5.4}$$

Similarly, the strain tensor may be decomposed in a spherical tensor $e = (\varepsilon_{kk}/3)\delta_{ij}$ and a deviator $e_{ij}$ as follows

$$\varepsilon_{ij} = \frac{\varepsilon_{kk}}{3} \delta_{ij} + e_{ij}. \tag{5.5}$$

Replacing $\sigma_{ij}$ and $\varepsilon_{ij}$ by their values (5.4) and (5.5) in Hooke's law (5.1) gives the relation

$$\frac{\sigma_{kk}}{3} \delta_{ij} + s_{ij} = (3\lambda + 2\mu) \frac{\varepsilon_{kk}}{3} \delta_{ij} + 2\mu e_{ij}. \tag{5.6}$$

As

$$3\lambda + 2\mu = 2\,G\,\frac{1+\nu}{1-2\nu} = 3K, \tag{5.7}$$

where

$$K = \frac{2G(1+\nu)}{3(1-2\nu)}\left[= \frac{E}{3(1-2\nu)}\right] \tag{5.8}$$

is the bulk modulus, equation (5.6) may be decomposed into

$$\begin{cases} \dfrac{\sigma_{kk}}{3} = 3K\,\dfrac{\varepsilon_{kk}}{3} \\ s_{ij} = 2Ge_{ij}. \end{cases} \tag{5.9}$$

In the literature, it is generally accepted that viscoelastic materials respond elastically to dilatations, but respond viscoelastically to distorsions. This means that the bulk modulus keeps its elastic value in creep deformations, whereas the Coulomb's modulus $G$ must be replaced by it Carson transform $\tilde{G}$.

Now, we can express the engineering constants $E$ and $\nu$ in terms of $K$ and $G$ as follows

$$E = \frac{9KG}{3K+G} \tag{5.10}$$

$$\nu = \frac{1}{2}\,\frac{3K-2G}{3K+G}. \tag{5.11}$$

We find therefore that the Carson transforms of $E$ and $\nu$ are

$$\tilde{E} = \frac{9K\tilde{G}}{3K+\tilde{G}} \tag{5.12}$$

and

$$\tilde{\nu} = \frac{1}{2}\,\frac{3K-2\tilde{G}}{3K+\tilde{G}} \tag{5.13}$$

respectively.

Finally, in the theory of plates, the flexural rigidity is given by the relation

$$D = \frac{Et^3}{12(1-\nu^2)} = \frac{2Gt^3}{12(1-\nu)} \tag{5.14}$$

where $t$ is the plate's thickness. Replacing $G$ by $\tilde{G}$ and $\nu$ by its $s$ transform $\tilde{\nu}$ given by (5.13), we find that

$$\tilde{D} = \frac{t^3}{12}\,\frac{2\tilde{G}}{1-\tilde{\nu}} = \frac{t^3}{12}\,\frac{2\tilde{G}(6K+2\tilde{G})}{3K+4\tilde{G}}. \tag{5.15}$$

We shall establish in the next paragraph the expression of the Carson transform of $G$

$$\tilde{G} = G\tilde{R} \tag{5.16}$$

and give the values of $\tilde{R}$ valid for the various mechanical models used in order to represent the distorsional behaviour of the viscoelastic material.

Replacing $\tilde{G}$ by its value (5.19) in formula (5.15), then dividing numerator and denominator by $4G$, gives

$$\tilde{D} = \frac{t^3}{12}\frac{2G\tilde{R}(6K + 2G\tilde{R})}{3K + 4G\tilde{R}} = \frac{Gt^3}{12}\frac{\tilde{R}\left(\frac{6K}{2G} + \tilde{R}\right)}{\frac{3K}{4G} + \tilde{R}} \tag{5.17}$$

Now, due to (5.8), we have

$$\frac{3K}{2G} = \frac{1 + \nu}{1 - 2\nu} = \rho \tag{5.18}$$

so that the expression of $\tilde{D}$ takes its final form

$$\tilde{D} = \frac{2Gt^3}{12}\frac{\tilde{R}(2\rho + \tilde{R})}{\rho + 2\tilde{R}}. \tag{5.19}$$

Therefore, in view of (5.14), we have

$$\frac{\tilde{D}}{D} = (1 - \nu)\frac{\tilde{R}(2\rho + \tilde{R})}{\rho + 2\tilde{R}}. \tag{5.20}$$

## 6. THE CLASSICAL VISCOELASTIC MODELS

It is generally accepted in the literature that the behaviour of the main viscoelastic materials employed in structural applications is represented with a sufficient approximation by the mechanical model obtained by coupling in series a Maxwell model with a Kelvin model. This so-called Boltzmann or Maxwell–Kelvin model is represented by Fig. 3. This

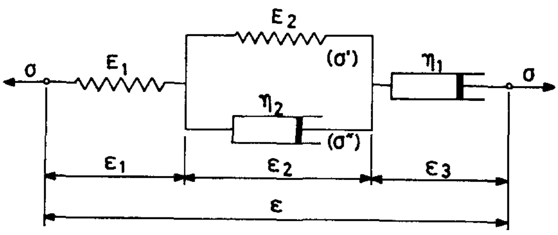


Fig. 3. The Boltzmann (or Maxwell–Kelvin) model.

model being consistently used in subsequent applications, it is useful to establish systematically its relaxation modulus $E(t)$ and its corresponding Carson transform $\tilde{E}$. The calculations are made systematically in following table:

| Constitutive equation | Laplace transform |
|---|---|
| $\varepsilon_1 = \dfrac{\sigma}{E_1}$ | $\bar{\varepsilon}_1 = \dfrac{\bar{\sigma}}{E_1}$ |
| $\varepsilon_2 = \dfrac{\sigma'}{E_2} \qquad \eta_2 \dfrac{d\varepsilon_2}{dt} = \sigma''$ | $\bar{\varepsilon}_2 = \dfrac{\bar{\sigma}'}{E_2}\ ; \quad s\eta_2\,\bar{\varepsilon}_2 = \bar{\sigma}''$ |
| $\sigma' + \sigma'' = \sigma$ | $\bar{\sigma}' + \bar{\sigma}'' = \bar{\sigma}$ |
| $E_2\varepsilon_2 + \eta_2 \dfrac{d\varepsilon_2}{dt} = \sigma$ | $(E_2 + s\eta_2)\bar{\varepsilon}_2 = \bar{\sigma}$ |
| $\eta_1 \dfrac{d\varepsilon_3}{dt} = \sigma$ | $s\eta_1\bar{\varepsilon}_3 = \bar{\sigma}$ |
| $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon$ | $\bar{\sigma}\left(\dfrac{1}{E_1} + \dfrac{1}{E_2 + s\eta_2} + \dfrac{1}{s\eta_1}\right) = \bar{\varepsilon}$ |

According to (5.7), $\bar{\sigma}/\bar{\varepsilon} = \tilde{E}(s)$. Thus,

$$\tilde{S}(s) = \frac{1}{E_1} + \frac{1}{E_2 + s\eta_2} + \frac{1}{s\eta_1} \tag{6.1}$$

which can be transformed into

$$\tilde{E}(s) = \frac{1}{\tilde{S}(s)} = \frac{E_1\left(s + \dfrac{E_2}{\eta_2}\right)s}{s^2 + \left(\dfrac{E_1}{\eta_1} + \dfrac{E_1}{\eta_2} + \dfrac{E_2}{\eta_2}\right)s + \dfrac{E_1E_2}{\eta_1\eta_2}}. \tag{6.2}$$

More generally, it is easily seen that, for the generalized Kelvin model of Fig. 4, the viscoelastic operator is given by the relation

$$\tilde{S} = \frac{1}{E_0} \sum_1^{n-1} \frac{1}{E_r + \eta_r s} + \frac{1}{\eta_n s} \tag{6.3}$$

whereas, for the generalized Maxwell model of Fig. 5,

$$\tilde{E} = \sum_{r=1}^{n} \frac{s}{\dfrac{s}{E_r} + \dfrac{1}{\eta_r}}. \tag{6.4}$$

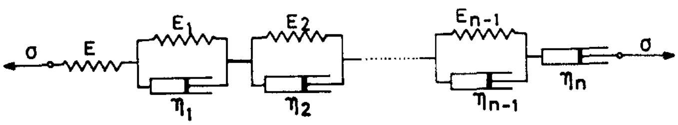


Fig. 4. Generalized Kelvin model.

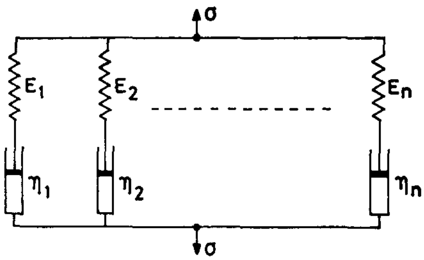


Fig. 5. Generalized Maxwell model.

Alternatively, equation (6.2) can be specialized to correspond to the usual degenerate cases:

Newton–Kelvin model: ($\varepsilon_1 = \infty$)
$$\tilde{E}(s) = \frac{(s + \varepsilon_2/\eta_2)s}{(1/\eta_1 + 1/\eta_2)s + E_2/\eta_1\eta_2}. \tag{6.5}$$

Standard linear solid: ($\eta_1 = \infty$)
$$\tilde{E}(s) = E_1 \frac{s + E_2/\eta_2}{s + (1/\eta_2)(E_1 + E_2)} \tag{6.6}$$
$$\tilde{S}(s) = \left(\frac{1}{E_1} + \frac{1}{E_2 + s\eta_2}\right) \tag{6.6 bis}$$

Kelvin model: ($E_1 = \eta_1 = \infty$)
$$\tilde{E}(s) = \eta(s + E_2/\eta_2) \tag{6.7}$$

Maxwell model ($E_2 = \eta_2 = \infty$)
$$\tilde{E}(s) = E_1 \frac{s}{s + E_1/\eta_1} \tag{6.8}$$
$$\tilde{S}(s) = \left(\frac{1}{E_1} + \frac{1}{\eta_1 s}\right) \tag{6.8 bis}$$

Viscous model ($E_1 = E_2 = \eta_2 = \infty$)
$$\tilde{E}_s = \eta_1 s. \tag{6.9}$$

The above expressions are valid for plane structures composed of bars, whose behaviour is uniquely governed by the Carson transform of Young's modulus $E$. In the case of space structures or plates and shells, above viscoelastic models represent the behaviour in shear and $E(s)$ should be replaced in all formulae by $\tilde{G}(s)$ and $E_1$ by $G$.

For example, in the particular case of a viscoelastic material which behaves in shear as a Maxwell material, we shall have

$$\tilde{G} = G \frac{s}{s + E_1/\eta_1}. \tag{6.10}$$

In general, we shall write

$$\tilde{G} = G\tilde{R}, \tag{6.11}$$

where $\tilde{R}$ is the rational fraction appearing behind $E_1$ in formulae (6.6), (6.8) and similar.

## 7. SOME SIMPLE EXAMPLES OF APPLICATIONS OF THE CLASSICAL CORRESPONDENCE PRINCIPLE

### 7.1 *Relaxation of a Maxwell material*

The constitutive equation is

$$\bar{\sigma} = \tilde{E}\,\bar{\varepsilon} \tag{7.1}$$

with

$$\tilde{E}(s) = E_1 \frac{s}{s + E_1/\eta_1}$$

A strain $\varepsilon\Delta(t)$, where $\Delta(t)$ is the unit step function, is suddenly applied at time $t = 0$. Find the relaxation law, that is the decrease of $\sigma$ with time. We replace in (7.1) $\tilde{E}$ by $s\bar{E}$, where $\bar{E}$ is given by (6.8). We find

$$\bar{\sigma} = \frac{E_1}{s + E_1/\eta_1}\,\bar{\varepsilon}$$

whose transform is

$$\sigma(t) = E_1\varepsilon\Delta(t)\exp[-(E_1/\eta_1)t] \tag{7.2}$$

### 7.2 *Creep of a Maxwell material*

We apply a sudden stress $\sigma = \sigma_0\,\Delta(t)$ at $t = 0$ (Fig. 6) and ask for the law $\sigma = \sigma(t)$. We write (7.1) in the form

$$\bar{\varepsilon} = \tilde{S}\bar{\sigma}$$

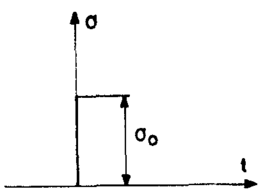


Fig. 6. Suddenly applied stress.

$\tilde{S}$ is given by $\dfrac{1}{E_1} + \dfrac{1}{\eta_1 s}$; thus, (7.3) becomes

$$\bar{\varepsilon} = \frac{1}{E_1}\,\frac{s + E_1/\eta_1}{s^2}\,\bar{\sigma}\,.$$

The Laplace transform is

$$\varepsilon = \frac{\sigma}{E_1}\,\Delta(t) + \frac{\sigma t}{\eta_1}$$

### 7.3 *Creep of the standard linear solid*

From (7.3) with $\tilde{S}$ given by (6.6 bis), we obtain

$$\bar{\varepsilon} = \left(\frac{1}{E_1} + \frac{1}{E_2 + s\eta_2}\right)\bar{\sigma}.$$

This may be written

$$\bar{\varepsilon} = \sigma_0\left(\frac{1}{E_1} + \frac{1}{E_2}\right)\left[\frac{1}{s(s + E_2/\eta_2)} + \frac{\dfrac{\eta_2}{E_1 + E_2}}{s + E_2/\eta_2}\right].$$

Applying the inverse Laplace transform, we find

$$\varepsilon = \sigma_0 \frac{E_2}{\eta_2}\left[\frac{\eta_2}{E_2}(1 - e^{-(E_2/\eta_2)t}) + \frac{\eta_2}{E_1 + E_2} e^{-(E_2/\eta_2)t}\right]$$

which may be simplified into

$$\varepsilon = \left[\sigma_0 \frac{E_1 + E_2}{E_1 E_2} - \sigma_0/E_2\, e^{-(E_2/\eta_2)t}\right] \Delta(t).$$

## 8. EXTENDED CORRESPONDENCE PRINCIPLE FOR IMPERFECT VISCOELASTIC STRUCTURES SUBJECT TO BUCKLING

The second variation of the strain energy, $U$, of an elastic structure depends linearly on the two elasticity moduluses, $E$ and $G$. In view of formulae (2.12) (or (2.13)), the eigenvalues $k_n$ (or $P^n_{cr}$) also depend linearly on $E$ and $G$. The same applies to the second variation of the strain energy, given by formula (2.21). Therefore, we obtain the *Extended correspondence principle* for structures *subject to buckling*:

*All the mathematical developments of Section* 3 *are valid for linearly viscoelastic structures, at the sole condition to replace all quantities f by their Laplace transforms* $\bar{f}$ *and the elasticity moduluses E and G by their s-multiplied transforms* (*also called Carson transforms*) $\tilde{E}$ *and* $\tilde{G}$.

We recall hereafter the main formulae of Section 3 and give their Laplace transforms. To establish the correspondence, we give the new formulae (8.8) and (8.10) the same numbers as in Section 3.

$$u_0 = \sum_{n=1}^{\infty} a_n^0 u_n \tag{3.1}$$

$$u = \sum_{n=1}^{\infty} \frac{1}{k_n/k - 1} a_n^0 u_n \tag{3.8}$$

$$F_0 = \sum_{n=1}^{\infty} F_n u_n \tag{3.3}$$

$$F = \sum_{n}^{\infty} \frac{1}{k_n/k - 1} F_n^0 u_n \tag{3.10}$$

$$\boxed{\bar{u} = \sum_{n=1}^{\infty} \frac{1}{1 - \bar{k}/\bar{k}_n} \overline{a_n^0}\, u_n} \tag{8.8}$$

$$\boxed{\bar{F} = \sum_{n=1}^{\infty} \frac{1}{\bar{k}_n/\bar{k} - 1} \overline{F_n^0}\, u_n} \tag{8.10}$$

## 9. SOME SIMPLE EXAMPLES OF THE EXTENDED CORRESPONDENCE PRINCIPLE TO AXIALLY LOADED BARS

### 9.1 *Simply supported bar, made of a Maxwell material possessing an imperfection affine to the first buckling mode*

The bar is represented by Fig. 7.

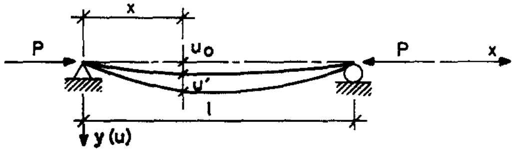


Fig. 7. Imperfect simply supported bar, axially compressed.

The initial deformation is

$$u_0 = f_0 \sin \frac{\pi x}{l}$$

and is affine to the first buckling mode

$$u_1 = \sin \frac{\pi x}{l}.$$

In this case, $k \equiv P$ and equation (3.8) reduces to

$$f = \frac{P}{P_1 - P} f_0.$$

The Laplace transform (8.8) of this equation reduces to

$$\bar{f} = \frac{P}{\bar{P}_1 - P} \bar{f}_0. \tag{9.1}$$

As the initial imperfection $f_0$ is supposed to be applied suddenly to the bar at the time $t = 0$, $f_0 = f_0 \Delta(t)$. Thus, its Laplace transform $\bar{f}_0$ is $f_0/s$. On the other hand, $\bar{P}_1 = P_1 (\tilde{E}/E_1)$, with

$$P_1 = \frac{\pi^2 E_1 I}{l^2}.$$

Replacing into equation (9.1), we obtain

$$\bar{f} = \frac{P}{P_1 \dfrac{\tilde{E}}{E_1} - P} \frac{f_0}{s} \tag{9.2}$$

Introduction of the non dimensional ratio

$$\alpha = \frac{P}{P_1}$$

and of the notation

$$\frac{\tilde{E}}{E_1} = \frac{A}{B} \tag{9.3}$$

yields

$$\bar{f} = \frac{\alpha}{\dfrac{A}{B} - \alpha} \frac{f_0}{s}. \tag{9.4}$$

According to Section 6,

$$\frac{\tilde{E}}{E_1} = \frac{A}{B} = \frac{s}{s + E_1/\eta_1}. \tag{6.8}$$

Introducing this value into (9.4) gives

$$\bar{f} = \frac{\alpha}{\dfrac{s}{s + \dfrac{E_1}{\eta_1}} - \alpha} \frac{f_0}{s}$$

which can be written

$$\bar{f} = \frac{f_0}{1-\alpha} \frac{s}{s\left(s - \dfrac{\alpha}{1-\alpha}\dfrac{E_1}{\eta_1}\right)} + \frac{\alpha}{1-\alpha}\frac{E_1}{\eta_1} f_0 \frac{1}{s - \dfrac{\alpha}{1-\alpha}\dfrac{E_1}{\eta_1}}.$$

According to the tables of Laplace transforms, the inverse transform of

$$\frac{1}{(s-a)(s-b)} \quad \text{is} \quad \frac{1}{a-b}(e^{at} - e^{bt}) \tag{9.5}$$

$$\frac{s}{(s-a)(s-b)} \quad \text{is} \quad \frac{1}{a-b}(ae^{at} - be^{bt}). \tag{9.6}$$

Transforming backwards expression (9.5) gives, after simplifications

$$f(t) = f_0\left\{-1 + \frac{1}{1-\alpha} \exp \frac{\alpha}{1-\alpha}\frac{E_1}{\eta_1} t\right\}. \tag{9.7}$$

9.2 *Same problem, but for a Maxwell–Kelvin material*

Instead of calculating $f(t)$, it is easier to calculate the percentage increase with time of the transverse displacement, i.e. the ratio

$$R = \frac{f(t) + f_0}{f(0) + f_0} \tag{9.8}$$

where $f(0)$ is the amplitude of the additional deflection taken by the compressed bar at time 0+, immediately after the initial imperfection $f_0$ has been applied.

Now, it is known by formula (3.9) reduced to its first term that

$$u_{\max} \equiv f(0) + f_0 = \frac{f_0}{1-\alpha}. \tag{9.9}$$

Therefore, the ratio $R$ to be determined reads

$$R = \frac{f(t) + f_0}{\dfrac{f_0}{1-\alpha}}.$$

Its Laplace transform is

$$\bar{R} = \frac{\bar{f} + \dfrac{f_0}{s}}{\dfrac{f_0}{1-\alpha}} \tag{9.10}$$

because, once again, the initial deflection is supposed to be applied as a step function of intensity $f_0$.

The developments of Section 9.1 are still valid up to formula (9.4), inclusive but $\bar{E}/E_1$ is now given by formula (6.2). Introducing expression (9.4) into (9.10) yields

$$\bar{R} = \frac{\dfrac{\alpha}{A/B - \alpha} + 1}{s\dfrac{f_0/s}{1-\alpha}}\frac{f_0}{s} = \frac{\dfrac{A}{B}(1-\alpha)}{s\left(\dfrac{A}{B} - \alpha\right)} = \frac{A(1-\alpha)}{s(A - \alpha B)}. \tag{9.11}$$

Formula (6.2) is equivalent to

$$A(s) = \left(s + \frac{E_2}{\eta_2}\right)s$$

$$B(s) = s^2 + \left(\frac{E_1}{\eta_1} + \frac{E_1}{\eta_2} + \frac{E_2}{\eta_2}\right)s + \frac{E_1 E_2}{\eta_1 \eta_2}.$$

With these values of $A$ and $B$, (9.11) becomes

$$\bar{R} = \frac{\left(s + \dfrac{E_2}{\eta_2}\right)}{s^2 + s\left[\dfrac{E_2}{\eta_2} - \dfrac{E_1}{1-\alpha}\left(\dfrac{1}{\eta_1} + \dfrac{1}{\eta_2}\right)\right] - \dfrac{\alpha}{1-\alpha}\dfrac{E_1 E_2}{\eta_1 \eta_2}}.$$

Introducing the simplifying notation

$$\beta = \frac{\alpha E_1}{1-\alpha}\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right), \tag{9.12}$$

we find that the roots of the denominator

$$s^2 + s\left(\frac{E_2}{\eta_2} - \beta\right) - \frac{\alpha}{1-\alpha}\frac{E_1 E_2}{\eta_1 \eta_2}.$$

are $n_1$, $n_2$, such that:

$$2n_{1,2} = \beta - \frac{E_2}{\eta_1} \pm \left[\left(\beta - \frac{E_2}{\eta_2}\right)^2 + \frac{4\alpha}{1-\alpha}\frac{E_1 E_2}{\eta_1 \eta_2}\right]^{1/2}. \tag{9.13}$$

We may therefore write

$$\bar{R} = \frac{s + \dfrac{E_2}{\eta_2}}{(s - n_1)(s - n_2)}. \tag{9.14}$$

Since, by (9.13)

$$n_1 + n_2 = \beta - \frac{E_2}{\eta_2},$$

formula (9.14) may still be written

$$\bar{R} = \frac{s + \beta - (n_1 + n_2)}{(s - n_1)(s - n_2)} = \frac{1}{n_1 - n_2}\left(\frac{\beta - n_2}{s - n_1} - \frac{\beta - n_1}{s - n_2}\right). \tag{9.15}$$

It is well known that the inverse transform of $\bar{R}$ is

$$R(t) \equiv \frac{f(t) + f_0}{f(0+)} = \frac{1}{n_1 - n_2}[(\beta - n_2)e^{n_1 t} - (\beta - n_1)e^{n_2 t}]. \tag{9.15 bis}$$

This result coincides with the analysis given by Kempner in [10].

9.3 *The most general solution for an axially compressed bar, made of any linearly viscoelastic material, having an arbitrary initial deformation $u_0 = u_0(x)$ and having arbitrary conditions of support at its ends*

To have a concrete problem in view, let us assume for example that the bar is simply supported at its left and built-in at its right end (Fig. 8.) The viscoelastic material composing the bar is represented by a model composed of any combination of springs and dash-pots. By the considerations of Section 6, it is possible to derive the rational fraction of the variable $s$

$$\frac{\bar{E}}{E_1} = \frac{A}{B}$$

representing the Laplace transform of the stress relaxation modulus. Considering the $n$th buckling mode $u_n = f_n \bar{u}_n(x)$, and the corresponding $n$th eigenvalue $k_n = P_{cr}^{(n)}$, we define the non-dimensional ratio

$$\alpha_n = \frac{P}{P_{cr}^{(n)}}.$$

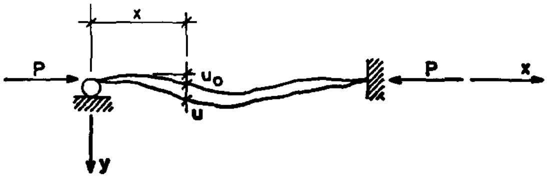


Fig. 8. Imperfect bar axially compressed, general case.

The percentage increase with time of the total transverse displacement corresponding to the $n$th mode $f_n(t)$ is

$$R_n(t) = \frac{f_n(t) + f_n^0}{f_n(0+) + f_n^0} = \frac{f_n(t) + f_n^0}{\dfrac{f_n^0}{1-\alpha}}. \tag{9.16}$$

Formula (9.11) is easily generalized for the $n$th mode into

$$\bar{R}_n = \frac{A(1-\alpha_n)}{s(A-\alpha_n B)} \tag{9.17}$$

and we can find by its inverse Laplace transform the function $R_n(t)$. $R_n(t)$ being known for all $n$, we have, by (9.16)

$$f_n(t) = \frac{f_n^0}{1-\alpha_n} R_n(t) - f_n^0. \tag{9.18}$$

The additional deformation in the $n$th mode is therefore

$$u_n(x, t) = f_n(t) u_n(x) = \left[\frac{f_n^0 R_n(t)}{1-\alpha_n} - f_n^0\right] u_n(x). \tag{9.19}$$

By the extended correspondence principle applied to the generalized principle of superposition (Sections 3 and 8), the variation with time of the additional deflection $u'$ is given by adding the contributions of the various modes, which gives finally

$$u(x, t) = \sum_{n=1}^{\infty} \left[\frac{f_n^0 R_n(t)}{1-\alpha_n} - f_n^0\right] u_n(x). \tag{9.20}$$

For example, in the concrete case of the built-in simply supported bar of Fig. 8, the eigenvalues are given by the formula

$$P_{cr}^{(n)} = \frac{k_n EI}{l^2} \tag{9.21}$$

where the $\sqrt{k_n}$ are the successive roots of the transcendental equation

$$\sqrt{k_n} = tg\sqrt{k_n}. \tag{9.22}$$

The first root of this equation is 4.493 and the higher order roots are given accurately enough by the formula

$$\sqrt{k_n} = \frac{2n+1}{2}\pi \tag{9.23}$$

On the other hand, the buckling modes are given by the general formula

$$u_n(x) = \sin\frac{x\sqrt{k_n}}{l} - \frac{x}{l}\sin\frac{\sqrt{k_n}}{l}. \tag{9.24}$$

## 10. EXAMPLE OF THE USE OF THE EXTENDED CORRESPONDENCE PRINCIPLE FOR A CASE OF FLEXURAL TORSIONAL BUCKLING

For the basic theory and the notations, we refer to the Appendix. To simplify foregoing calculations, we restrict ourselves to the problem of flexural torsional buckling of a bar whose cross section is symmetrical with respect to $Gz$ (Fig. 9). Then, $y_s = 0$ and the differential equations of the perfect bar (equations (15) of Appendix) reduce to

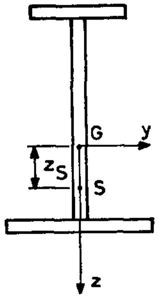


Fig. 9. Monosymmetrical cross-section.

$$\begin{cases} EI_z v'''' + Pv'' - Pz_s\psi'' = 0 \\ EI_y w'''' + Pw'' = 0 \\ EI\psi'''' - GJ\psi'' + Pi_p^2\psi'' - Pz_s v'' = 0. \end{cases} \tag{10.1}$$

It is visible that they decouple into an Eulerian buckling in the $Gz$ plane, governed by second equation (10.1) and a flexural torsional buckling in the $Gy$ plane, governed by the system of equations

$$\begin{cases} EI_z v'''' + Pv'' - Pz_s\psi'' = 0 \\ EI\psi'''' - (GJ - Pi_p^2)\psi'' - Pz_s v'' = 0. \end{cases} \tag{10.2}$$

In what follows, we assume that the bar is simply supported and that warping is free at both ends, which gives the following boundary conditions

$$v = v'' = 0, \; \psi = \psi'' = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = l. \tag{10.3}$$

In that case, it is well known (see e.g. |2|, p. 226) that the buckling modes are

$$v_n = A_n b \sin\frac{n\pi x}{l}, \quad \psi_n = B_n \sin\frac{n\pi x}{l}. \tag{10.4}$$

In these expressions, $b$ is whatever dimension of the cross section, introduced in order to render coefficients $A_n$, $B_n$, nondimensional. To simplify further calculations, we introduce the notations

$$P_y^n = \frac{n^2\pi^2 EI_z}{l^2}, \qquad P_x^n = \frac{1}{i_p^2}\left(GJ + E\Gamma\frac{n^2\pi^2}{l^2}\right), \tag{10.5}$$

which represent the $n$th buckling loads for eulerian buckling in the $Gy$ plane and pure torsional buckling, respectively. Introducing now expression (10.4) into the differential equations (10.2) and using notations (10.5), we find

$$\begin{cases} P_y^n A_n b - PA_n b + Pz_s B_n = 0 \\ Pz_s A_n b + i_p^2(P_x^n - P)B_n = 0. \end{cases} \tag{10.6}$$

These algebraic equations are linear and homogeneous in $A_n$, $B_n$. Buckling ($A_n$, $B_n \neq$ !) only occurs if these equations are compatible, which requires that

$$\begin{vmatrix} (P_y^n - P)b & Pz_s \\ Pz_s b & i_p^2(P_x^n - P) \end{vmatrix} = 0. \tag{10.7}$$

From (10.7), we find the classical expression of the flexural torsional buckling load of order $n$

$$P_{cr}^{(n)} = \frac{P_x^n + P_y^n + \sqrt{(P_x^n - P_y^n)^2 + 4P_x^n P_y^n(z_s^2/i_p^2)}}{2(1 - z_s^2/i_p^2)} \qquad (<P_x < P_y). \tag{10.8}$$

We can normalize the buckling modes by expressing the normalizing condition (20) (see Appendix). This reduces here to

$$\int_0^1 (v_n'^2 - 2z_s v_n' \psi_n' + i_p^2 \psi_n'^2)\, dx = 1 \quad \text{for all } n. \tag{10.9}$$

Replacing $v_n$ and $\psi_n$ by their expressions (10.4), one finds

$$\frac{n^2\pi^2}{2l}(A_n^2 b^2 - 2z_s b A_n B_n + i_p^2 B_n^2) = 1 \quad \text{for all } n. \tag{10.10}$$

This condition added to one of the conditions (10.6), for example the first one:

$$(P_y^n - P_{cr}^n) b A_n + P_{cr}^n z_s B_n = 0, \tag{10.11}$$

determines completely the values of $A_n$ and $B_n$ for all $n$ and therefore the amplitudes of the buckling modes.

We consider now an elastic bar possessing some initial imperfections $v_0(x)$, $\psi_0(x)$. In line with the results of the Appendix, we call $u_0(x)$ a scalar quantity representing symbolically these initial imperfections. If these imperfections can be expanded into series of the buckling modes of the form

$$v_0(x) = \sum_{n=1}^{\infty} a_n^0 v_n(x), \quad \psi_0(x) = \sum_{n=1}^{\infty} a_n^0 \psi_n(x), \tag{10.12}$$

then we can take for example $u_0 \equiv v_0$ and represent the imperfect shape by†

$$u_0 = \sum_{n=1}^{\infty} a_n^0 u_n(x). \tag{10.13}$$

According to Section 3—which has been shown in the Appendix to be applicable to cases of spatial buckling—the additional deformation of the imperfect bar when the compression force is applied is

$$u(x) = \sum_{n=1}^{\infty} \frac{1}{P_n/P - 1} a_n^0 u_n(x). \tag{10.14}$$

In particular, if the initial imperfection is affine to the first buckling mode† (10.13)

† The first buckling mode is given by (10.4) as

$$v_1(x) = A_1 b \sin\frac{\pi x}{l}, \quad \psi_1(x) = B_1 \sin\frac{\pi x}{l},$$

If we put $A_1 b = f$ and $B_1/A_1 b = K$, these expressions may be written

$$v_1(x) = f \sin\frac{\pi x}{l}, \quad \psi_1(x) = Kf \sin\frac{\pi x}{l}. \tag{10.15}$$

where $K$ is completely determined by (10.10) and (10.11). The assumption made means exactly that the initial imperfection is given by:

$$v_0(x) = f_0 \sin\frac{\pi x}{l}, \quad \psi_0(x) = Kf_0 \sin\frac{\pi x}{l} \tag{10.16}$$

reduces to

$$u_0(x) = a_1^0 u_1(x) = f_0 u_1(x) \tag{10.17}$$

and the amplitude of the additional deflection is

$$f = \frac{P}{P_1 - P} f_0 . \tag{10.18}$$

We suppose now that the imperfect bar studied hereabove is made of the linearly viscoelastic material studied in Sections 5 and 6. Because of the extended correspondence principle developed in Section 8, the behaviour with time of this bar is governed by the Laplace transform of (10.18)

$$\bar{f} = \frac{P}{\bar{P}_1 - P} \bar{f}_0 \tag{10.19}$$

with $\bar{f}_0 = f_0/s$, which has the same form (9.1) as in Section 9. However, here, the behaviour of the bar with time will entirely depend on the constitutive equations of the viscoelastic material.

*First case.* If the viscoelastic material shows constant Poisson's ratio $\nu$ under viscoelastic deformation—as it is approximately the case for aged concrete—the viscoelastic moduluses $\tilde{E}$ and $\tilde{G}$ remain proportional to each other because

$$\tilde{G} = \frac{\tilde{E}}{2(1+\nu)} . \tag{10.20}$$

In this case, equations (10.8) and (10.5) written for $n = 1$ show that

$$\bar{P}_1 = P_1 \frac{\tilde{E}}{E_1} \tag{10.21}$$

(as in Section 9), where $P_1$ is the critical load of flexural torsional buckling of the *corresponding elastic* bar (given by (10.8) for $n = 1$). The mathematical developments are therefore identical to those obtained in Sections 9.1 and 9.2 for Eulerian buckling and the final results (9.7) and (9.15) are valid here.†

*Second case.* On the contrary, if the viscoelastic material responds elastically to dilatation, but viscoelastically to distorsions, as discussed in Section 5, then we must, in order to obtain the Laplace transform of (10.19), take account of the fact that, by (10.5), $P_y^1$ is proportional to $E$ and $P_x^1$ involves both $E$ and $G$, and that the first flexural torsional buckling load $P_1$ (given by (10.18) for $n = 1$) is a complicated combination of $P_x^1$ and $P_y^1$. Following par. 5, we must therefore replace everywhere $G$ by $G\tilde{R}$ and $E$ by :

$$\tilde{E} = \frac{9K\tilde{G}}{3K + \tilde{G}} = \frac{9KG\tilde{R}}{3K + G\tilde{R}} , \tag{10.22}$$

† This result can be shown to apply to other cases of spatial instability, as for instance lateral buckling.

We then find that $\bar{P}_1/P_1$ is a complicated function of $\tilde{R}$ that we call $F(\tilde{R})$. Transforming (10.19) as in Section 9.1. and putting $\alpha = P/P_1$, we come to a relation of the form

$$\bar{f} = \frac{\alpha}{F(\tilde{R}) - \alpha} \frac{f_0}{s}. \tag{10.23}$$

There are no difficulties other than algebraic ones to find the inverse Laplace transform of (10.23):

$$f = f(t), \tag{10.24}$$

which describes the additional deformation of the compressed bar in the course of time.

## 11. EXAMPLE OF THE USE OF THE EXTENDED CORRESPONDENCE PRINCIPLE FOR CASES OF BUCKLING OF PLATES

Consider the following problem: a rectangular plate (Fig. 10) made of a linearly visco-elastic material, is simply supported on its four edges. It is uniformly compressed in the horizontal direction. Find its behaviour along the time. It is supposed that the initial deflection $w_0$ of the plate

$$w_0(x, y) = f^0_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{11.1}$$

is affine to the $(m, n)$ buckling mode

$$w_{mn}(x, y) = f \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{11.2}$$

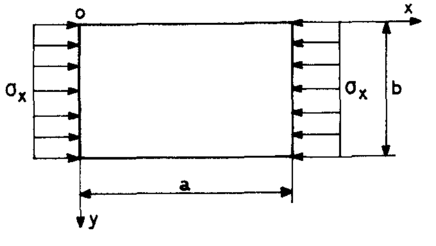


Fig. 10. Simply supported rectangular plate, axially compressed.

The buckling critical stresses (eigenvalues) are given by the formula

$$\sigma^{mn}_{cr} = k_{mn} \sigma_E \tag{11.3}$$

where

$$\sigma_E = \frac{\pi^2 D}{b^2 t} \tag{11.4}$$

is a reference stress called Euler stress,

$$k_{mn} = \left(\frac{bm}{a} + \frac{na}{bm}\right)^2 = \left(\frac{m}{\alpha} + \frac{n\alpha}{m}\right)^2 \tag{11.5}$$

is the (non-dimensional) buckling coefficient and

$$\alpha = \frac{a}{b} \tag{11.6}$$

is the aspect ratio of the plate.

The Laplace transform of (10.3) is

$$\bar{\sigma}_{cr}^{mn} = k_{mn}\bar{\sigma}_E \tag{11.7}$$

and, taking account of formulae (11.4) and (5.19), the Laplace transform of $\sigma_E$ becomes

$$\bar{\sigma}_E = \frac{\pi^2 \bar{D}}{b^2 t} = \frac{\pi^2}{b^2 t}\frac{2Gt^3}{12}\frac{\tilde{R}(2\rho + \tilde{R})}{\rho + 2\tilde{R}}. \tag{11.8}$$

It follows from Section 8—and especially formula (8.2)—that the Laplace transform of the additional deflexion $f$ is:

$$\bar{f} = \frac{f_{mn}^0/s}{\bar{K}_{mn}/K - 1} \tag{11.9}$$

where $\bar{K}_{mn} \equiv \bar{\sigma}_{cr}^{mn}$ and $K = \sigma = \dfrac{N_x}{t}$. Taking account of (11.7) and (11.8), we obtain

$$\bar{f}_{mn} = \frac{\dfrac{f_{mn}^0}{s}}{\dfrac{1}{N_x}k_{mn}\dfrac{\pi^2}{b^2}\dfrac{2Gt^3}{12}\dfrac{\tilde{R}(2 + \tilde{R})}{\rho + 2\tilde{R}} - 1}. \tag{11.10}$$

A formula similar to (11.10) was obtained by Lin [13] by a rather different reasoning and for the particular case of a viscoelastic material which behaves under deviatoric stresses as a Maxwell material. In that case, by Section 6,

$$\tilde{R} = \frac{s}{s + \beta} \quad \text{with} \quad \beta = \frac{E_1}{\eta_1}.$$

Lin gives in his paper the corresponding value of $f_{mn}^{(t)}$ obtained by applying the inverse Laplace transform to (11.10), as well as a complete numerical example. The interest of the present derivation is to show that the result (11.10) is completely general in the sense that it applies to any loading case like compression, shear, bending, or combinations of these loadings.

As soon as the function $f_{mn}(t)$ is found, the deflection of a plate possessing the initial deflection

$$w_0(x, y) = \sum_m \sum_n f_{mn}^0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{11.11}$$

is obtained. Indeed, by the extended superposition principle,

$$w(x, y, t) = \sum_m \sum_n f_{mn}(t) \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{11.12}$$

This result can still be generalized for plates of any shape and any boundary conditions by replacing the double sine series by a series in terms of the buckling modes of the considered plate.

## APPENDIX

*Generalization of the properties of imperfect elastic structures to a case of spatial instability: Flexural–torsional buckling of an axially compressed bar*

1. *Variational formulation of the problem for the perfect bar.* As was mentioned by Washizu, [3], p. 313, it is possible to formulate the variational problem of the flexural torsional buckling of an axially compressed bar which is clamped at one end ($x = 0$) and is subjected to an axial force $P_w$ at the other end ($x = 1$) by introducing the approximate displacement field:

$$\begin{aligned} U(x, y, z) &= u - yv'_G - zw'_G + \psi'\varphi(y, z) \\ V(x, y, z) &= v_G - y(1 - \cos\psi) - z\sin\psi \\ W(x, y, z) &= w_G + y\sin\psi - z(1 - \cos\psi), \end{aligned} \tag{1}$$

where $U$, $V$, $W$, are the components of the displacement of an arbitrary point $(y, z)$ of the bar, $u(x)$, $v_G(x)$ and $w_G(x)$, functions of $x$ only, are the components of the displacement of a point $G$ of its axis. $\psi$ is the angle of torsion and $\varphi(y, z)$ is Saint-Venant's warping function. The $y$ and $z$ axes are taken to coincide with the principal axes through the centroïd of the cross section.

According to Trefftz [11, 12], the coordinates of the shear center of the cross-section (which is also the center of twist), are

$$y_s = -\frac{1}{I_y}\iint z\varphi\, dy\, dz, \quad z_s = \frac{1}{I_z}\iint y\varphi\, dy\, dz \tag{2}$$

with

$$I_y = \iint z^2\, dy\, dz \text{ and } I_z = \iint y^2\, dy\, dz; \tag{3}$$

and the torsional rigidity is $C = GJ$ with

$$J = \iint \left(\frac{d\varphi}{dy} z - \frac{d\varphi}{dz} y + y^2 + z^2\right) dy\, dz. \tag{4}$$

Saint-Venant's warping function is chosen so that

$$\iint \varphi\, dy\, dz = 0. \tag{5}$$

Trefftz has also shown that, if $\varphi_s(y, z)$ is the Saint-Venant warping function with the shear axis as the axis of rotation and is so chosen that

$$\iint \varphi_s(y_s\, z)\, dy\, dz = 0, \tag{6}$$

we have

$$\varphi_s(y, z) = \varphi(y, z) - z_s y + y_s z \tag{7}$$

$$\Gamma_s = \Gamma - y_s^2 I_y - z_s^2 I_z \tag{8}$$

where

$$\Gamma = \iint \varphi^2(y, z)\, dy\, dz \quad \text{and} \quad \Gamma_s = \iint \varphi_s^2(y, z)\, dy\, dz \tag{9}$$

are the warping rigidities of the cross-section corresponding to the points $G$ (centroïd) and $S$ (shear center) respectively. For practical calculations, it is more convenient to take as variables the displacements $v$, $w$ of the shear center $S$. As is well known, the relations between $v_G$, $w_G$ and $v$, $w$, are as follows:

$$v = v_G + z_s\psi, \qquad w = w_G - y_s\psi.$$

Either by above method or by the technical theory of thin-walled bars with open cross-section, [2], its is possible to show that

$$U = \frac{1}{2}\int_0^1 (EI_y w''^2 + EI_z v''^2 + GJ\psi'^2 + E\Gamma\psi''^2)\,dx \tag{10}$$

$$V = -\frac{P}{2}\int_0^1 \{(v'^2 + w'^2) + i_p^2\psi'^2 + 2y_s w'\psi' - 2z_s v'\psi'\}\,dx \tag{11}$$

with the notations

$$i_p^2 = y_s^2 + z_s^2 + I_p/A, \qquad I_p = \iint (y^2 + z^2)\,dy\,dz, \qquad A = \iint dy,\,dz. \tag{12}$$

$\Pi = U + V$ is the variation of total potential energy during buckling and, if the bar is in indifferent equilibrium, we must have (see e.g. [2]) for the true buckling displacements $v$, $w$, $\psi$:

$$\Pi = (U + V) = 0 = \text{minimum}. \tag{13}$$

The extreme condition (13) on $\Pi$ requires that the variations of this quantity with respect to the three independent displacements be equal to zero:

$$\delta_v(\Pi) = 0, \qquad \delta_w(\Pi) = 0, \qquad \delta_\psi(\Pi) = 0. \tag{14}$$

This yields the following system of differential equations:

$$\begin{cases} EI_z v'''' + Pv'' - Pz_s\psi'' = 0 \\ EI_y w'''' + Pw'' + Py_s\psi'' = 0 \\ E\Gamma\psi'''' - GJ\psi'' + Pi_p^2\psi'' - Pz_s v'' + Py_s w'' = 0. \end{cases} \tag{15}$$

It can be shown (see [2]) that this system has non trivial solutions $(v_r, w_r, \psi_r)$ called buckling modes, for a discrete series $P_1, P_2, \ldots, P_n, \ldots$ of parameter $P$, called eigenvalues or buckling loads. Let us now consider the displacement field

$$v = v_r + \varepsilon V, \qquad w = w_r + \varepsilon W, \qquad \psi = \psi_r + \varepsilon\Psi \tag{16}$$

where $(v_r, w_r, \psi_r)$ is the buckling mode number $r$, $\varepsilon$ a small number, and $(V, W, \Psi)$ an admissible displacement field, satisfying the end geometrical boundary conditions on $v$, $w$, $\psi$.

Reasoning as in [2], pp. 144–147, we obtain the variational equation associated with present buckling problem:

$$\begin{aligned} &\int_0^1 \{EI_y w_r''W'' + EI_z v_r''V'' + GJ\psi_r'\Psi' + E\Gamma\psi_r''\Psi''\}\,dx \\ &\qquad - P_r\int_0^1 \{v_r'V' + w_r'W' + i_p^2\psi_r'\Psi' + y_s(w_r'\Psi' + W'\psi_r') \\ &\qquad - z_s(v_r'\Psi' + V'\psi_r')\}\,dx = 0 \qquad (r = 1, 2, \ldots, \infty), \end{aligned} \tag{17}$$

which is valid for all buckling modes and all admissible fields $(V, W, \Psi)$†.

† Calling symbolically $u_r$ the $r$th buckling mode $(v_r, w_r, \psi_r)$ and $\xi$ the admissible displacement field $(V, W, \Psi)$, we can write (17) in the form

$$V_{11}(u_r, \xi) - \lambda_r T_{11}(u_r, \xi) = 0,$$

where $V_{11}(a, b)$ and $T_{11}(a, b)$ are the bilinear functionals associated to the quadratic and homogeneous functionals $V$ and $T$.

Applying this variational equation with $V = v_s$, $W = w_s$, $\Psi = \psi_s$, then writing it again for index $s$ and applying it with $V = v_r$, $W = w_r$, $\Psi = \psi_r$, then subtracting both equations obtained and noting that $P_s \neq P_r$, we obtain the two orthogonality conditions connecting the buckling modes

$$\begin{cases} T_{11}(u_r, u_s) \equiv \int_0^1 [v_r' v_s' + w_r' w_s' + i_p^2(\psi_r' \psi_s') - y_s(w_r' \psi_s' + w_s' \psi_r') \\ \qquad + z_s(v_r' \psi_s' + v_s' \psi_r')] \, dx = 0 \quad (s \neq r) \\ U_{11}(u_r, u_s) \equiv \int_0^1 [EI_y w_r'' w_s'' + EI_z v_r'' v_s'' + E\Gamma \psi_r'' \psi_s'' + GJ \psi_r' \psi_s'] \, dx = 0 \quad (s \neq r). \end{cases} \tag{18}$$

It is convenient to normalize the buckling modes so that (see, 2.12)

$$T(u_r) = -\frac{1}{k_r} U(u_r) = 1 \quad \text{for all } r. \tag{19}$$

In present case, this normalizing condition becomes:

$$T(v_r, w_r, \psi_r) \equiv \int_0^1 [v_r'^2 + w_r'^2 + 2y_s w_r' \psi_r' - 2z_s v_r' \psi_r' + i_p^2 \psi_r'^2] \, dx = 1 \quad \text{for all } r \tag{20}$$

whence Rayleigh quotient (2.13) gives

$$U(v_r, w_r, \psi_r) = P \,. \tag{21}$$

If $v(x)$, $w(x)$, $\psi(x)$ are given deflexion and rotation functions of the bar, all of them satisfying the given geometrical end boundary conditions, we may expand these functions as series in terms of the buckling functions $v_n(x)$, $w_n(x)$, $\psi_n(x)$ in the form:

$$v(x) = \sum_{n=1}^{\infty} a_n v_n(x); \quad w(x) = \sum_{n=1}^{\infty} b_n w_n(x); \quad \psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x). \tag{22}$$

In what follows, we shall restrict ourselves to the consideration of deflexions for which $a_n = b_n = c_n$. In this case,

$$v(x) = \sum_{n=1}^{\infty} a_n v_n(x); \quad w(x) = \sum_{n=1}^{\infty} a_n w_n(x); \quad \psi(x) = \sum_{n=1}^{\infty} a_n \psi_n(x). \tag{23}$$

The coefficients $a_n$ are given by the formula

$$a_n = \int_0^1 [v' v_n' + w' w_n' + i_p^2 \psi' \psi_n' - y_s(w' \psi +_n' w_n' \psi') + z_s(v' \psi_n' + v_n' \psi')] \, dx. \tag{24}$$

Indeed, if we replace in the right hand member of (24) $v$, $w$, and $\psi$ by their expressions (23), then put the summation sign $\sum_{n=1}^{\infty}$ before the integral sign, we see that, by virtue of (18), all integrals $T_{11}(u_r, u_n) = 0$, except $T_{11}(u_n, u_n) \equiv T(u_n)$ which is equal to 1 by (20).

2. *Variational formulation of the behavior of an imperfect bar.* In paragraph 1 of this Appendix, we have considered a perfect bar in indifferent equilibrium. Let us now consider an imperfect bar possessing small initial deviations $v_0(x)$, $w_0(x)$, $\psi_0(x)$ from the perfect shape and subjected to a compression force $P < P_{cr}$. This bar is in stable equilibrium and, therefore, its total potential energy

$$\Pi = U + V \tag{25}$$

is minimum for the correct configuration.

The expressions of the strain energy $U$ and the potential energy of the external force $P$, $V$, of the imperfect bar may be easily deduced from the corresponding expressions (10) and (11) for the perfectly straight bar. Indeed, if $v(x)$, $w(x)$ and $\psi(x)$ represent now the additional displacements due to the imposition of the thrust $P$, the expression (10) of $U$ remains valid. On the other hand, in order to evaluate the potential energy of $P$, we proceed like in paragraph 3 and give first to the imperfect bar negative displacements $-v_0$, $-w_0$, $-\psi_0$, in order to bring it back to its straight shape, and then give to this "model" the total displacements $v_0 + v$, $w_0 + w$, $\psi_0 + \psi$; in this way, we find for the imperfect structure

$$\begin{aligned} V = -\frac{P}{2}\int_0^1 \{&(v' + v_0')^2 + (w' + w_0')^2 + 2y_s[(w' + w_0')(\psi' + \psi_0')] \\ &+ 2z_s[(v' + v_0')(\psi' + \psi_0')] + i_p^2(\psi' + \psi_0')^2 \\ &- v_0'^2 - w_0'^2 - 2y_s w_0' \psi_0' + 2z_s v_0' \psi_0' - i_p^2 \psi_0'^2\}\, \mathrm{d}x. \end{aligned} \tag{26}$$

Suppose now that the initial deflexions of the bar are expanded in the series of the buckling modes given by (23):

$$v_0(x) = \sum_{n=1}^{\infty} a_n^0 v_n(x); \quad w_0(x) = \sum_{n=1}^{\infty} a_n^0 w_n(x); \quad \psi_0(x) = \sum_{n=1}^{\infty} a_n^0 \psi_n(x). \tag{27}$$

Let the additional displacements which the bar takes upon application of compression force $P$ be expressed by similar series:

$$v(x) = \sum_{n=1}^{\infty} a_n v_n(x); \quad w(x) = \sum_{n=1}^{\infty} a_n w_n(x); \quad \psi(x) = \sum_{n=1}^{\infty} a_n \psi_n(x). \tag{28}$$

Introducing expansions (27) and (28) in the expressions (10), (26) of $U$ and $V$ and taking account of the orthogonality relations (18) as well as the normality conditions (20) (21), we find:

$$U = \sum_{n=1}^{\infty} P_n a_n^2 \tag{29}$$

$$V = -P \sum_{n=1}^{\infty} [(a_n^0 + a_n)^2 - a_n^2]. \tag{30}$$

These expressions of $U$ and $V$ are identical to the expressions (3.4) and (3.5) of Section 3. This shows that the results of that paragraph are still applicable when the displacement field $(v, w, \psi)$ has more than one component and therefore vectorial character, at the sole condition to designate by $u_0$ and $u$, the common amplitude of the three components of the initial deviations from straightness $(v_0, w_0, \psi_0)$ and of the additional deformations under load $(v, w, \psi)$, respectively.

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**Резюме** — В прежней работе автор доказывал, что поведение несовершенных эластичных структур подвергнутых критической силе, вызывающей потерю устойчивости, можно предсказать на основании собственных значений и собственных колебаний. После краткого перечня этих характеристик, во-первых показано — в приложении — что они простираются на такие случаи пространственного изгиба, как искривление и скручивание. Во-вторых, что принцип соответствия действительный для поведения первого порядка линейно-упруговязких структур можно широко обобщить — использованием преобразования ЛАПЛАСА — к проблемам коробления каких бы то не было конституэнтов уравнения материалов.

И наконец, детально рассматривали несколько примеров Эйлерового изгиба или изгибно-крутильного коробления бруска и пластины.